

What Can One Learn About Self-Organized Criticality from Dynamical Systems Theory?

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We develop a dynamical system approach for the Zhang model of self-organized criticality, for which the dynamics can be described either in terms of iterated function systems or as a piecewise hyperbolic dynamical system of skew-product type. In this setting we describe the SOC attractor, and discuss its fractal structure. We show how the Lyapunov exponents, the Hausdorff dimensions, and the system size are related to the probability distribution of the avalanche size via the Ledrappier–Young formula.

KEY WORDS: Self-organized criticality; hyperbolic dynamical systems; iterated functions systems.

I. INTRODUCTION

Within the last 10 years the notion of Self-Organized Criticality (SOC) became a new paradigm for the explanation of a huge variety of phenomena in nature and social sciences. Its origin lies in the attempt to explain the widespread appearance of power-law like statistics for characteristic events in a multitude of examples like the distribution of the size of earthquakes, $1/f$ -noise, amplitudes of solar flares, species extinction... to name only a very few cases.^(1–3, 19) As a result, an important literature in physics has been devoted to the study of systems exhibiting SOC.

The complexity of the dynamics in the above mentioned systems is mainly due to the presence of long-range spatial and time correlations, leading to non trivial effects like anomalous diffusion. At stationarity, the

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average incoming flux of external perturbations is simply compensated by the average outgoing flux that can leave the system at the boundary, or by dissipation in the bulk. Therefore, there is a constant flux through the system, leading to a nonequilibrium situation. What is remarkable in this stationary state, referred to as the SOC state, is that the distribution of avalanches appears to follow a power law, namely there is scale invariance reminiscent of thermodynamic systems at the critical point. This is certainly one central reason why SOC has attracted the physicist community: these systems (apparently) reach *spontaneously* a critical state without any fine tuning of some control parameter.

Several models have been proposed to mimic these mechanisms like the sandpile model,^(1,2) the abelian sandpile⁽¹²⁾ or the continuous energy model.⁽³⁰⁾ Numerical simulations on one hand, and theoretical approaches on the other hand have led to a good description of SOC, in particular with respect to the computation of critical exponents that are believed to characterize the universality class the model belongs to, as they do in second order phase transitions.

However, to our knowledge, no serious attempt has been made to study SOC from a dynamical system point of view (except refs. 7 and 11). It is however a natural approach to try to access the macroscopic behaviour of large sized systems from the microscopic dynamical evolution. The macroscopic behaviour at stationarity is characterized by a probability measure one has to extract from the microscopic evolution. One is seeking a “good” measure from a physical point of view, namely a Sinai–Bowen–Ruelle measure (SBR): in the SOC model we discuss later this measure maximizes the entropy.

In this paper we develop a dynamical system description for a certain class of SOC models (like the Zhang model⁽³⁰⁾), for which the whole SOC dynamics can either be described in terms of Iterated Function Systems, or as a piecewise hyperbolic dynamical system of skew-product type where one coordinate encodes the sequence of activations. Several deep results from the theory of hyperbolic dynamical systems can then be used, having interesting implications on the SOC dynamics, provided one makes some natural assumption (like ergodicity) which will be partially justified in this paper.

With this approach we give a precise definition of the SOC attractor discussed by some people.^(1,2) We show that it has a fractal structure for low values of the critical energy. The main objects for which our point of view is appropriate is certainly the structure of the asymptotic energy distribution or, in other words, the structure of the natural invariant measure. We show in particular how the Lyapunov exponents, the geometric structure of the support of the invariant measure (Hausdorff dimensions), and

the system size are related to the probability distribution of the avalanche size, via the Ledrappier–Young formula.⁽²⁰⁾

II. THE DYNAMICAL STRUCTURE OF THE ZHANG MODEL

A. Description of the Model

In this paper we deal with the Zhang model on a d dimensional, connected subgraph $A \subset \mathbb{Z}^d$, with nearest neighbours edges, though the formalism we develop holds for more general graphs. Let ∂A be the boundary of A , namely the set of points in \mathbb{Z}^d/A at distance 1 from A and let N be the cardinality of A . Each site $i \in A$ is characterized by its “energy” X_i , which is a non-negative real number. The “state” of the network is completely defined by the configuration of energies $\mathbf{X} = \{X_i\}_{i \in A}$. Let E_c be a real, positive number, called the critical energy, and $\mathcal{M} = [0, E_c[{}^N$. Let $d_{1,2}(\mathbf{X}, \mathbf{Y})$ be the L_1 (resp. L_2) distance on \mathcal{M} . A configuration \mathbf{X} is “stable” iff $\mathbf{X} \in \mathcal{M}$ and “unstable” or “overcritical” otherwise. If \mathbf{X} is stable then we choose a site i at random with probability $1/N$ and add to it energy δX . As far as the physically relevant parameter is the local rigidity $E_c/\delta X$,⁽²³⁾ one can investigate the cases where E_c varies, and where δX is a constant. We will therefore assume that $\delta X = 1$ without loss of generality. If a site i is overcritical ($X_i \geq E_c$), it loses a part of its energy in equal parts to its $2d$ neighbours. Namely, we fix a parameter $\varepsilon \in [0, 1[$ such that, after relaxation of the site i , the remaining energy of i is εX_i , while the $2d$ neighbours receive the energy $((1-\varepsilon)X_i)/2d$. Note that in the original Zhang model,⁽³⁰⁾ ε was taken to be zero. We define here a straightforward extension. Note however that in this paper ε will be considered as a *small* parameter compared to E_c .

If several nodes are simultaneously overcritical, the local distribution rules are additively superposed, i.e., the time evolution of the system is synchronous. The sites of ∂A have always zero energy (dissipation at the boundaries). The succession of updating leading an unstable configuration to a stable one is called an *avalanche*. Because of the dissipation at the boundaries, all avalanches are *finite*. The structure of an avalanche can be encoded by the sequence of overcritical sites $A = \{A_i\}_{0 \leq i}$ where $A_0 = \{a\}$, the activated site, and $A_i = \{j \in A \mid X_j \geq E_c \text{ in the } i\text{th step of avalanche}\}$, $i > 0$.

The addition of energy is *adiabatic*. When an avalanche occurs, one waits until it stops before adding a new energy quantum. Further activations eventually generate a new avalanche, but, because of the adiabatic rule, each new avalanche starts from *only one* overcritical site.

Since the avalanche after activation of site a maps overcritical to stable configurations one can view this process as a mapping from $\mathcal{M} \rightarrow \mathcal{M}$ where one includes the process of activation of site a . We hence associate a map T_a with the activation at vertex a . This map usually has singularities and therefore different domains of continuity denoted below by \mathcal{M}_a^k where k runs through a finite set depending on a . Call $T_a^k = T_a|_{\mathcal{M}_a^k}$. The main object of this paper is the study of the properties of the family of mappings $\{T_a^k\}$ and to link these properties to the asymptotic behaviour.

B. Piecewise Affine Mappings

1. Structure of the Piecewise Affine Mappings. One can easily write the conditions on the stable energy configurations insuring that the avalanche $A = \{A_i\}_{0 \leq i}$ occurs. This defines a convex domain⁴ in \mathcal{M}_a^k in \mathcal{M} . The \mathcal{M}_a^k 's are the domains of continuity of T_a , and they constitute, for each a , a partition of \mathcal{M} . There is therefore a one to one correspondence between an avalanche and a map T_a^k . The energy distributions rules of Zhang model implies that:

$$T_a^k \cdot \mathbf{X} = L_a^k \cdot (\mathbf{X} + \mathbf{e}_a) = L_a^k \cdot \mathbf{X} + L_a^k \cdot \mathbf{e}_a, \quad \mathbf{X} \in \mathcal{M}_a^k \quad (1)$$

where the *linear* mapping L_a^k characterizes the redistribution of energies on each site after the avalanche. The *column* i of L_a^k 's contains the ratios of energy given by the site j to the other sites after the corresponding avalanche. Alternatively, the entries of the *row* i correspond to the energy received by i from the others sites (i included). \mathbf{e}_a being the canonical basis vector of \mathbb{R}^N in the direction corresponding to the activation at site a , the constant vector $L_a^k \cdot \mathbf{e}_a$ corresponds to the redistribution of the additional energy $\delta X = 1$ on each site, after the avalanche. In the case where no relaxation occurs the corresponding map is just a shift along the a axis. A way to build L_a^k is to construct it step by step, by a left product of elementary matrices giving the redistribution of energy from one step in the avalanche to the successive step. The composition of these matrices is determined by the avalanche profile.

Let $\mathcal{S}_a^k = \partial \mathcal{M}_a^k$. Then $\mathcal{S}_a = \bigcup_k \mathcal{S}_a^k$ is the set of singularities for the transformation T_a . The sets \mathcal{S}_a^k are unions of segments of hyperplanes in \mathbb{R}^N .

2. Protection Effect. The original Zhang model contains a pathology. Due to the reset to zero of an overcritical site after relaxation

⁴ For example, the domain of energy such that a site a do not relax after activation is delimited by the hyperplane $E_a = E_c - 1$, and the boundaries of \mathcal{M} .

($\varepsilon = 0$), linear dependencies among the sites (more precisely, direction of \mathbb{R}^N associated to these sites) are created along the avalanche. This implies the existence of a non trivial kernel. Thus each L_a^k 's is a *projection* onto a subspace of \mathbb{R}^N , whose dimension increases with the number of involved sites in the avalanches. The one step matrices have a number of zero eigenvalues given by the number of sites set to zero at the corresponding time step. Multiplying these matrices gives raise to the kernel of L_a^k . We denote the subspace generated by zero eigenvalues $\mathcal{E}^0(a, k)$. Note that, in general, $\text{Ker } L_a^k \subset \mathcal{E}^0(a, k)$. Clearly, the existence of a nontrivial kernel is the source of several mathematical complications when studying the dynamics of Zhang model. It is a very particular feature of the $\varepsilon = 0$ model. It makes however the global geometry of the attractor quite interesting (see Fig. 1).

3. Contraction. Each mapping L_a^k has only eigenvalues of modulus lower or equal than 1. Indeed, by definition, the sites of the boundaries of the avalanche receive energy without relaxation. This implies that, by eventually permuting the basis vectors, the linear map L can be written as:

$$L = \begin{bmatrix} I & * \dots * & 0 \\ 0 & \begin{bmatrix} & & \\ & & \\ & * \dots * & \end{bmatrix} & 0 \\ 0 & * \dots * & I \end{bmatrix} \quad (2)$$

where I is the identity matrix and where the *'s can be zero or not. They correspond to the fraction of energy received by the sites which have not relaxed. Therefore, the vectors corresponding to sites not relaxing are eigenvectors of L with eigenvalue one. We denote the corresponding (neutral) subspace by $\mathcal{E}^n(a, k)$.

On the other hand, the inner block in (2) corresponds to the sites which have relaxed. The energy conservation implies that the sum on each column of the block is strictly lower than one (some part of the energy has gone outside the block, to the sites on the boundary of the avalanche). By usual arguments on positive matrices it follows that the eigenvalues are *strictly lower than one* in the block.⁽¹⁵⁾ Note that, for $\varepsilon = 0$, this block contains also the subspace $\mathcal{E}^0(a, k)$. Therefore, the subspace of relaxing sites is decomposed into two subspaces: $\mathcal{E}^0(a, k)$ and $\mathcal{E}^-(a, k)$, where $\mathcal{E}^-(a, k)$ denotes the subspace associated to the eigenvalues $0 < |\lambda_i| < 1$.

Hence with each mapping L_a^k we associate the following decomposition:

$$\mathcal{E}^-(a, k) \oplus \mathcal{E}^n(a, k) \oplus \mathcal{E}^0(a, k) = \mathbb{R}^N \quad (3)$$

C. Composed Mapping

1. Composition of Affine Mappings. Extended Dynamical System. The activation dynamics can be represented by the *left Bernoulli shift* σ over Σ_A^+ , the set of right infinite sequence $\mathbf{a} = \{a_1, \dots, a_k, \dots\}$, $a_k \in A$, where $\sigma\mathbf{a} = a_2 a_3 \dots$. Namely, a_n is the n th activated site in the activation sequence \mathbf{a} . We denote by $[a]$ the set of sequences whose first digit is a .

The combined effect of the activation and relaxation process is than described by a dynamical system of skew-product type $\mathcal{T}: \Omega \rightarrow \Omega$ such that:

$$\mathcal{T}(\hat{\mathbf{X}}) \stackrel{\text{def}}{=} (\sigma\mathbf{a}, T_{a_1}(\mathbf{X})); \quad \hat{\mathbf{X}} \stackrel{\text{def}}{=} (a, \mathbf{X})$$

where $\Omega = \Sigma_A^+ \times \mathcal{M}$ is called the *extended phase space*.

Let $D\mathcal{T}_{\hat{\mathbf{X}}}$ be the tangent map of \mathcal{T} at $\hat{\mathbf{X}}$. (When speaking about differentials of \mathcal{T} we usually think of Σ_A represented by a smooth system $z \rightarrow |A| \cdot z \bmod 1$).

The singularity set of \mathcal{T} is:

$$\mathcal{S} = \bigcup_{a \in A} [a] \times \mathcal{S}_a \quad (4)$$

We define a distance on Ω by $d_\Omega(\hat{\mathbf{X}}, \hat{\mathbf{Y}}) = d_{\Sigma_A^+}(\mathbf{a}, \mathbf{a}') + d_{\mathcal{M}}(\mathbf{X}, \mathbf{Y})$, where $\hat{\mathbf{X}} = (\mathbf{a}, \mathbf{X})$, $\hat{\mathbf{Y}} = (\mathbf{a}', \mathbf{Y})$. We denote the two projections on the first and second coordinate by $\pi^u(\mathbf{a}, \mathbf{X}) = \mathbf{a}$, and $\pi^s(\mathbf{a}, \mathbf{X}) = \mathbf{X}$. The superscript u, s means respectively *unstable* and *stable* and will be explained below. We have a natural partition of Ω , $\mathcal{P} = \{\mathcal{P}_a^k = [a] \times \mathcal{M}_a^k\}$. Note that \mathcal{P} is a generating partition for (\mathcal{T}, Ω) in the topological sense, that is, the diameter of the elements of $\bigvee_i \mathcal{T}^{-i}\mathcal{P}$ goes to zero. Furthermore the \mathcal{M}_a^k are convex.

2. Kernel of the Infinite Product Map. For $\varepsilon = 0$ the kernel of the map \mathcal{T}^t can increase with t , projecting \mathbb{R}^N onto spaces of lower and lower dimensions. Therefore, after a certain, *finite time*, $n(\hat{\mathbf{X}})$, $\hat{\mathbf{X}}$ is projected onto the effective lower dimensional subspace,^{5, 6}

$$\mathcal{E}^s(\hat{\mathbf{X}}) \stackrel{\text{def}}{=} \{\mathbf{v} \in \{0\} \times \mathbb{R}^N; \forall t \geq 0, \|D\mathcal{T}_{\hat{\mathbf{X}}}^t \cdot \mathbf{v}\| > 0\} \quad (5)$$

⁵ One has to check that this space is not finally reduced to $\{0\}$! However, this would imply that all vectors in \mathbb{R}^N are asymptotically mapped to 0. This is excluded since the avalanche matrices have non negative entries.

⁶ Note that the dimension of this space depends *a priori* on $\hat{\mathbf{X}}$. However, if ergodicity holds, this dimension is constant for (almost-every) initial condition.

It is somehow the reference space with respect to $\hat{\mathbf{X}}$, because, asymptotically, the dynamics of vectors in \mathbb{R}^N under $D\mathcal{T}_{\hat{\mathbf{X}}}$ reduces to the dynamics of vectors initially in $\mathcal{E}^s(\hat{\mathbf{X}})$. We get therefore a splitting of the projection on \mathcal{M} of the tangent space at $\hat{\mathbf{X}}$ as:

$$\mathbb{R}^N = \mathcal{E}^s(\hat{\mathbf{X}}) \oplus \mathcal{H}(\hat{\mathbf{X}}) \quad (6)$$

where $\mathcal{H}(\hat{\mathbf{X}})$ is the kernel of the product map. This splitting will be refined further below by using the Oseledec space decomposition.

3. Local Contraction. One easy shows that for any finite connected A and arbitrary activation sequence \mathbf{a} all sites become overcritical infinitely often. Assume the opposite. Then, there exists a site which is overcritical only a finite number of times along the infinite sequence \mathbf{a} but has a neighbour site which relaxes infinitely often. The energy coming from the overcritical neighbour site is larger than $((1 - \varepsilon) E_c)/2d$ by definition. This implies that all neighbours relax also an infinite number of times during the whole sequence. Hence we get a contradiction. It follows that there exists a time $\tau \equiv \tau(A, E_c, \varepsilon) < \infty$ such that, $\forall \hat{\mathbf{X}}$, after at most τ time steps each site has been at least once overcritical. By looking at the product map $D\mathcal{T}_{\hat{\mathbf{X}}}^\tau$ this implies that all eigenvalues are different from one. This is straightforward since the sum of entries on each column of the composed map on \mathcal{M} is strictly lower than one (and is bounded away from 1). Therefore there is a positive constant $C \equiv C(A, E_c, \varepsilon)$ s.t.:

$$\|\pi^s D\mathcal{T}_{\hat{\mathbf{X}}}^\tau\|_1 = \sup_{\|V\|_1=1} \|\pi^s D\mathcal{T}_{\hat{\mathbf{X}}}^\tau V\|_1 < C < 1 \quad (7)$$

This implies that the map \mathcal{T}^τ acts as local contraction in all directions in the space \mathcal{M} , along the trajectory of any point $\hat{\mathbf{X}}$. This has in particular, the following consequence. The distance of two points $\hat{\mathbf{X}}, \hat{\mathbf{Y}}$ whose trajectory belong to the same domain of continuity eventually goes to zero if the trajectories lie in the same domains of continuity along the whole activation sequence.

4. Hyperbolic Structure and Lyapunov Exponents. Assume that almost every point is regular, namely the map \mathcal{T} is differentiable along all points of the trajectory (note that as long as we work in the tangent spaces this assumption is not necessary since the involved mapping are all well defined at ∂P_i . Only for the construction of the local induced stable manifolds in \mathcal{M} one has to take care of regularity). Then, one can decompose the (effective) tangent space at a.e. point $\hat{\mathbf{X}} \in \Omega$ into a contracting subspace $\mathcal{E}^s(\hat{\mathbf{X}})$ and an expanding one $\mathcal{E}^u(\hat{\mathbf{X}})$, s.t.:

1. $\forall \hat{\mathbf{X}}, \mathcal{E}^s(\hat{\mathbf{X}}) \oplus \mathcal{E}^u(\hat{\mathbf{X}}) = \mathbb{R}^{N+1 - \dim(\mathcal{H}(\hat{\mathbf{X}}))}$.
2. $\forall \hat{\mathbf{X}}, \exists \lambda < 1, \tau < \infty$, s.t. $\|D\mathcal{T}_{\hat{\mathbf{X}}}^\tau\|_{\mathcal{E}^s(\hat{\mathbf{X}})} \leq \lambda$. Furthermore $\|D\mathcal{T}_{\hat{\mathbf{X}}}\|_{\mathcal{E}^u(\hat{\mathbf{X}})} = N = |A|$.
3. $\mathcal{T}(\mathcal{E}^s(\hat{\mathbf{X}})) = \mathcal{E}^s(\mathcal{T}(\hat{\mathbf{X}})); \mathcal{T}(\mathcal{E}^u(\hat{\mathbf{X}})) = \mathcal{E}^u(\mathcal{T}(\hat{\mathbf{X}}))$

Furthermore $\mathcal{E}^s(\hat{\mathbf{X}})$ can be decomposed into a sequence of subspaces,^(14, 24, 29) ⁷

$$\mathcal{E}^s(\hat{\mathbf{X}}) = \mathcal{E}_1(\hat{\mathbf{X}}) \supset \mathcal{E}_2(\hat{\mathbf{X}}) \supset \dots \supset \mathcal{E}_l(\hat{\mathbf{X}}) \quad (8)$$

such that if $\mathbf{v} \in \mathcal{E}_i(\hat{\mathbf{X}}) \setminus \mathcal{E}_{i+1}(\hat{\mathbf{X}})$ the average contraction of \mathbf{v} is given by the Lyapunov exponent:

$$\lambda_i(\hat{\mathbf{X}}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\mathcal{T}_{\hat{\mathbf{X}}}^n(\mathbf{v})\|_2 \quad (9)$$

From property (7) there are no zero Lyapunov exponents (note however that some exponents go to zero as E_c tends to infinity).

If the dynamics is ergodic the Lyapunov exponents are almost-surely constants and the same holds for $\dim \mathcal{E}_i$. Corresponding to the shift action there is a *positive* Lyapunov exponent, which is trivially $\log(N)$. The Lyapunov exponents are directly related to the geometrical structure of the support of the invariant measure. In the Zhang model the negative exponents $0 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ are physically related to the transport of energy and the dissipation rate at the boundary.⁽⁹⁾ In particular, we show below that there is a natural relation linking these exponents to the avalanche size distribution. The positive Lyapunov exponent $\lambda_0 = \log N$ corresponds to the entropy production coming from the activation dynamics.

The average exponential volume contraction rate on $\mathcal{E}^s(\hat{\mathbf{X}})$ is given by the sum:

$$\sum_{i=1}^N \lambda_i \quad (10)$$

while the average exponential variation rate of the volume in the extended phase space is $\log N + \sum_{i=1}^N \lambda_i$.

⁷ In fact, in the $\varepsilon = 0$ case we have still to assume that the angle between $\mathcal{E}^s(\hat{\mathbf{X}})$ and $\mathcal{H}(\hat{\mathbf{X}})$ is bounded away from zero, a.s. because, otherwise, Lyapunov exponents might not exist.

For regular $\hat{\mathbf{X}}$ let

$$\mathcal{W}_\varepsilon^s(\hat{\mathbf{X}}) = \{ \hat{\mathbf{Y}} : d(\hat{\mathbf{X}}, \hat{\mathbf{Y}}) \leq \varepsilon, \forall i \geq 0,$$

$\mathcal{T}^i \hat{\mathbf{X}}$ and $\mathcal{T}^i \hat{\mathbf{Y}}$ are in the same partition element of \mathcal{P} }

be the ε local stable manifold. Clearly one has on $\mathcal{W}_\varepsilon^s$ uniform exponential contraction.

The global stable manifold $\mathcal{W}^s(\hat{\mathbf{X}})$ is obtained by

$$\bigcup_{i \geq 0} \mathcal{T}^{-i}(\mathcal{W}_\varepsilon^s(\hat{\mathbf{X}}))$$

Finally let $\mathcal{W}_{\text{loc}}(\hat{\mathbf{X}})$ be the largest connected component of \mathcal{W}^s containing $\hat{\mathbf{X}}$. Since the system is of skew product type one has a trivial unstable manifold being in the case of representing the shift as $z \rightarrow z \cdot |A| \bmod 1$ the whole interval $[0, 1]$. Note that $\mathcal{W}_\varepsilon^s(\hat{\mathbf{X}})$ may not exist if $\exists \{n_i\}$ s.t. $d(\mathcal{T}^{n_i}(\hat{\mathbf{X}}), \mathcal{S}) < e^{-n_i C}$ where $C > 0$ is some constant larger than $-\lambda$. The set of points with this property has measure zero unless the invariant measure concentrates on \mathcal{S} . This aspect will be described in more detail in ref. 8. We make the following conjecture:

Conjecture 1. There exists a $\bar{E}_c(N)$, such that, for Lebesgue almost-every $E_c < \bar{E}_c(N)$ there exists an $n(E_c, N)$ and a ν such that $\forall t > n(E_c, N)$:

$$d(\mathcal{T}^t(\Omega), \mathcal{S}) > \nu > 0$$

This implies that after a finite time the dynamics stays away from the singularity set. This assumption is sufficient for the existence of local stable manifolds, but it will have several other important implications. We expect Conjecture 1 to be true for \bar{E}_c sufficiently small since for $\bar{E}_c \ll 1$ the contraction dominates the expansion in the extended phase space. The invariant set has the structure of a totally disconnected Cantor set with large gaps. Furthermore the local structure of this invariant set is constant for open sets of E_c values (see Section 6), but the singularity set is varying continuously in E_c (except for a countable set), one can find open domains of E_c values where \mathcal{S} stays away from the invariant set.

The singularity set has nevertheless the following effect on the dynamics. Take an η -ball of initial conditions (in \mathcal{M}), and fix an activation sequence. For η large enough the image of the ball under some iterate of \mathcal{T} is cut by the singularity set. This means that the points separated by the singularity set will not evolve under the same sequence of mappings. For large size systems this can cause on \mathcal{M} a kind of expansion effect (for fixed typical activation sequence) on a mesoscopic scale.

5. Symbolic Dynamics. Symbolic dynamics is a very useful tool for the investigation of the orbit structure of dynamical systems. To this aim, one fixes a partition \mathcal{P} of the phase space and associates to each point the sequence of partition elements visited by the orbit of a point. To make symbolic dynamics useful one wants this correspondence essentially to be unique (that is up to sets of measure zero). Furthermore to handle the symbolic dynamics it is essential to have an explicit characterization of the legal (that is by orbits generated) set of symbolic sequences. The perhaps most prominent example of such an explicit description are symbolic systems defined by a Markov transition graph called Subshift Of Finite Type (SFT). A classical result in hyperbolic dynamics says that uniform hyperbolic systems are always conjugated to SFT.^(10, 25, 27) The specific partitions giving rise to such coding are called Markov Partitions.

In the Zhang model one can encode the possible transitions between avalanches in a *transition graph* with respect to the canonical partition $\mathcal{P} = \{\mathcal{P}_a^k\}$ from Section 5. Namely, we draw an arrow from \mathcal{P}_a^k to \mathcal{P}_b^l if and only if $\mathcal{T}(\mathcal{P}_a^k) \cap \mathcal{P}_b^l \neq \emptyset$. We denote by $\Sigma_{\mathcal{P}}^+$ the set of admissible infinite sequences w.r. to the partition \mathcal{P} .

Clearly, points on $W_{\text{loc}}^s(\hat{\mathbf{X}})$ form an equivalence class for the symbolic coding induced by \mathcal{P} . Note that the transition graph is a priori Markov only for special choices of E_c .

When the invariant set is bounded away from the singularity set one can refine the partition $\{\mathcal{P}_a^k\}$ to make it Markov (this is certainly not a necessary assumption to get a Markov transition graph). Namely, there is an m s.t. the partition $\bigvee_{i=1}^m \mathcal{T}^{-i}(\mathcal{T}^m \mathcal{P})$ is a Markov partition. We label the affine mappings corresponding to the Markov partition elements by F_i . Note that several F_i can usually correspond to the same map T_a^k .

Let us give an example. In the case $E_c \in [1, 2]$, $\varepsilon = 0$, in one dimension, the piecewise continuous mapping applied is uniquely determined by the position of the zero site. Indeed, after a sufficiently long sequence all sites have energy $X_i \geq E_c/2$ but eventually one with a zero value. Since a site with value zero is the only possible stopping site for an avalanche besides the boundary the avalanche is *uniquely determined* by the position of the activated site and of the zero. This case is however the simplest, because there is no need to cut further the \mathcal{P}_a^k 's in order to get a SFT. For $E_c > 2$ things are more complicated, due to the presence of sites with integer values $1 \dots [E_c - 1]$ which may stop an avalanche, according to the amount of energy they receive.

As already said we expect the above mentioned property of being disjoint from the singularities to hold for a.e. E_c value less than some $\bar{E}_c(N)$. This implies that the system is a SFT. For $\varepsilon = 0$, $d = 1$ there is another dense set of E_c values for which one can show that the system is a SFT.

Proposition 1. For $\varepsilon = 0$, $d = 1$ and $E_c = n/(2d)^p$ for any $n, p \in \mathbb{N}^*$ the system is conjugate to a SFT.

Note that the elementary operations on each avalanche and each node i are of the form $X_i \rightarrow X_i + \sum_j X_j/2$ for some j and a check whether X_i is larger or less than E_c . If E_c is of the above form it is a finite digit number (eventually zero) in base 2. It follows that for each point $\mathbf{X} \in \mathcal{M}$ one has to know only a uniformly bounded, finite number of digits in base 2 to decide in which set $\mathcal{M}_a^k \mathbf{X}$ is. The same holds for the legal transitions between avalanches domains, that is, there is a finite number of forbidden strings in base $2^{|A|}$ coding the whole system, hence it is a SFT. ■

6. Macroscopic State and SBR Measures. The addition of energy on one hand, and the dissipation of exceeding energy at the boundaries, on the other hand, drives gradually the system towards a stationary state where there is a constant energy flux through the system. As far as our representation accounts for activation dynamics on one hand and transport-dissipation (avalanche) on the other hand, the full informations about the macroscopic behaviour of the system at stationarity is contained in the *invariant measures*⁸ of our dynamical system. Since Ω has a product structure one has canonical measures μ^u (induced measure on the unstable direction) and μ^s (induced measure on \mathcal{M}). For simplicity we will assume that μ is a Bernoulli measure, namely that the sites are chosen independently with fixed rates. Once we have fixed the distribution of activation, we are interested on the possible μ^s measures. Of special physical importance are the measures obtained by iterating the Lebesgue measure μ_L on \mathcal{M} , that is $\lim_{n \rightarrow \infty} (1/n) \sum_{i=0}^{n-1} \mathcal{T}^i(\mu^u \times \mu_L)$. We call this measure conditional SRB with respect to μ^u .

It is common in the SOC literature to assume ergodicity. In our setting the physically relevant ergodic property is equivalent to the following conjecture.

Conjecture 2. For any E_c , A , ε , and given μ^u the corresponding conditional SBR measure is unique.

This implies in particular the almost-sure equality between the ensemble average and the time average for typical energy configurations. We give some arguments to support this assumption at least for certain E_c values.

⁸ Namely $\mu(\mathcal{T}^{-1}(\mathcal{B})) = \mu(\mathcal{B})$ where \mathcal{B} is a measurable set in Ω .

If the probability of activation of any site is non zero then there exists a periodic point $\hat{\mathbf{X}}$ with period p such that

$$\mu^u([\pi^u(\hat{\mathbf{X}}), \pi^u(\mathcal{T}(\hat{\mathbf{X}})), \dots, \pi^u(\mathcal{T}^{p-1}(\hat{\mathbf{X}}))] > 0 \quad (11)$$

where $[\pi^u(\hat{\mathbf{X}}), \pi^u(\mathcal{T}(\hat{\mathbf{X}})), \dots, \pi^u(\mathcal{T}^{p-1}(\hat{\mathbf{X}}))]$ is a cylinder set, i.e. this is the set of infinite sequences in $\Sigma_{\mathcal{A}}^+$ which coincide with the activation sequence of $\hat{\mathbf{X}}$ on the p first symbols. Assume that the periodic orbit admits a stable manifold such that

$$\bigcup_i \mathcal{W}^s(\mathcal{T}^i(\hat{\mathbf{X}})) = \mathcal{M} \quad (12)$$

If the periodic point does not lie on \mathcal{S} one can take a small neighbourhood $\mathcal{U}_\varepsilon(\hat{\mathbf{X}})$. Otherwise, as far as the singularity set is moving with E_c while the limit cycle does not change on open domains of E_c (the maps remain the same) one can change E_c by an arbitrary small value in order to make the limit cycle disjoint from \mathcal{S} . Due to (11), a generic sequence \mathbf{a} admits arbitrary long segments with repeated words $\pi^u(\hat{\mathbf{X}}), \pi^u(\mathcal{T}(\hat{\mathbf{X}})), \dots, \pi^u(\mathcal{T}^{p-1}(\hat{\mathbf{X}}))$. Therefore, from (12), almost every point visits $\mathcal{U}_\varepsilon(\hat{\mathbf{X}})$ infinitely often for any $\varepsilon > 0$ sufficiently small. By the hyperbolic structure and some moderate assumptions on the distribution of the size of $\mathcal{W}_{\text{loc}}^s(\hat{\mathbf{X}})$ one can then form a Hopf chain⁹ between iterates of a.e. points $\hat{\mathbf{Z}}, \hat{\mathbf{Y}}$ when they visit $\mathcal{U}_\varepsilon(\hat{\mathbf{X}})$. By standard arguments from ergodic theory concerning the equality of forward and backward averages one can then prove that a.e. pair of points on the invariant set belongs to the same ergodic component. In general, it does not seem easy to give explicit examples of sequences of avalanches satisfying the above conditions. The crucial point here is to show (12). But perhaps it should be possible to weaken the above conditions substantially.

For $d = 1$, $E_c > 1$, one can check it by using the following argument. For $E_c > 1$, starting from any stable configuration, one can add energy to the low energy sites ($E_i < E_c - 1$) in order to get a configuration where all sites have energy $E_c - 1 < E_i < E_c$. Activating any site in this configuration generates a unique “maximal” avalanche where all sites become overcritical. This avalanche is recurrent¹⁰ and there exists a periodic orbit satisfying (12). For $d > 1$ the number of reflexions of the front on the boundaries can vary and there are several types of “maximal” avalanches which makes the argument break. One can however still apply it on a diamond shaped lattice with L odd ($N = L^2$), because, by activating periodically in the middle

⁹ A path made of pieces of local stable and unstable manifolds.

¹⁰ This argument has been already used by other authors like Dhar,⁽¹²⁾ and Speer⁽²⁸⁾ for the Dhar model.

site, one has essentially the same situation as for the one dimensional chain. For E_c small, especially for $E_c < 1$ the avalanche patterns are much more complicated and the above argument breaks down.

Note that in the case where we have a Markov graph, the ergodic property can in principle be directly checked on the Markov transition graph defined in the previous section. Namely, if the Markov transition graph is asymptotically irreducible and aperiodic, (unique) ergodicity follows from usual results on Markov chains.

We proceed in discussing some aspects of the dependence of μ on E_c for fixed A .

Proposition 2. μ^s is singular for all E_c sufficiently small.

Proof. This follows easily since for $E_c \ll 1$ one can make the L_1 norm of all avalanche map arbitrary small since the avalanche has to “reflect” many times on each boundary node, hence every node has contributed to the dissipation. Since the expansion is constant it follows that $\det(D\mathcal{T}) < 1$ hence all measures are singular. ■

Proposition 3. μ^s is atomic for the chain and $E_c \in [(1 + \varepsilon)/(1 - \varepsilon), 2/(1 - \varepsilon)]$.

This is proved in Section III.A.3. Furthermore, we conjecture the following:

Conjecture 3. The Hausdorff dimension of μ^s is piecewise continuous and monotonously increasing on the domains of continuity for $E_c \ll 1$.

This is supported by the following argument. On open intervals I_i of E_c the structure of the mappings T_a^k does not change but the domains of continuity \mathcal{M}_a^k do. Furthermore for E_c decreasing the probabilities for avalanches with higher contraction should increase which should force the Hausdorff dimension to increase monotonously with E_c on each I_i .

We now discuss the connection between the invariant measure and the SOC state. It is possible to extract from μ the probability distribution of all observables usually considered in the study of SOC. The traditionally used observables are: the *duration* t (number of iteration steps inside one avalanche); the *size* s (total number of relaxing sites counted with multiplicity), and the *area* a (number of *distinct* relaxing sites). Fix now an observable, say s . Let \mathcal{H}_s be the set of mappings T_a^k with avalanche size s and let \mathcal{Q}_s be the union of its domains \mathcal{M}_a^k . Let $P_N(s)$ be the probability

to have an avalanche of size s for a lattice of size N in the stationary limit. One has clearly:

$$P_N(s) = \mu^s(\mathcal{Q}_s) \quad (13)$$

7. Ledrappier–Young Formula. This formula plays a key role in relating the probability of avalanche size to the average contraction rate (sum of Lyapunov exponents). It establishes a kind of conservation law relating the Lyapunov exponents, some version of Hausdorff dimension and the Kolmogorov–Sinai entropy.

One can refine the foliation into a stable and unstable manifold by splitting the manifolds into sub-manifolds $\mathcal{W}_i^s(\hat{\mathbf{X}})$ (resp. $\mathcal{W}_i^u(\hat{\mathbf{X}})$) such that the contraction (resp. the expansion) on $\mathcal{W}_i^s(\hat{\mathbf{X}})$ (resp. $\mathcal{W}_i^u(\hat{\mathbf{X}})$) is governed by the Lyapunov exponent λ_i . Let δ_i be the local Hausdorff dimension of the measure μ projected on $\mathcal{W}_i^r(\hat{\mathbf{X}})$ (where r stands for s, u), namely:

$$\delta_i = \lim_{\varepsilon \rightarrow 0} \frac{\log \mu(\mathcal{B}_i(\hat{\mathbf{X}}, \varepsilon))}{\log \varepsilon} \quad (14)$$

where $\mathcal{B}_i(\hat{\mathbf{X}}, \varepsilon)$ is an ε -ball around $\hat{\mathbf{X}}$ in $\mathcal{W}_i^r(\hat{\mathbf{X}})$. Then $\sigma_i = \delta_i - \delta_{i+1}$, $i = 1 \dots N-1$ is the transverse dimension of the measure μ on $\mathcal{W}_i^r(\hat{\mathbf{X}}) \setminus \mathcal{W}_{i+1}^r(\hat{\mathbf{X}})$. It is constant for μ almost-every $\hat{\mathbf{X}}$ if μ is ergodic. The unstable foliation being one dimensional in our context, the Hausdorff dimension of the measure $\mathcal{W}^u(\hat{\mathbf{X}})$ is δ_0 . It is equal to one for the uniform activation measure.

Let h_μ be the Kolmogorov–Sinai entropy of μ and λ_i^+ the positive Lyapunov exponents. The Ledrappier–Young formula is ref. 20 (for ergodic measures):

$$h_\mu = \sum_i \lambda_i^+ \sigma_i \quad (15)$$

where the sum is taken over the positive Lyapunov exponents. It expresses in particular that without any absolute continuity of any equation relating entropy and positive Lyapunov exponents must involve some notion of fractional dimension. In our case, it reduces to:

$$h_\mu = \log N \delta_0 \quad (16)$$

From now on we will assume that $\mu^u = \mu_L$ (uniform activation). In this case $\delta_0 = 1$.

When the dynamics is invertible, this formula, applied to the inverted system, gives the following equality in the Zhang model:

$$\sum_{i=1}^N \lambda_i \sigma_i = -\log N \tag{17}$$

where the sum is now taken over the negative Lyapunov exponents.

However, one has to assume that the dynamics is (μ almost-surely) invertible. That means physically that, at stationarity, the probability that two avalanches, starting from two different configurations, end on the same configuration of energies is zero. Like conjecture 1 we expect this property to hold only for small E_c values, where the invariant set is a Cantor set but to fail generically for large E_c values.

We have the following conjecture:

Conjecture 4. For E_c sufficiently small, there exists a $n(E_c, N, d)$ such that $\forall t > n(E_c, N)$:

$$\mu(\mathcal{T}^t(\mathcal{P}_i^k) \cap \mathcal{T}^t(\mathcal{P}_j^l)) = 0, \quad \forall \mathcal{P}_i^k \neq \mathcal{P}_j^l$$

Note that one can weaken this assumption by requiring that there are, on the attractor, less than N preimages and still get a nontrivial relation to Lyapunov exponents. One can still write down a Ledrappier–Young formula for non invertible systems by making the system invertible⁽²⁶⁾ by coding the backward iteration tree in the same way as we did with the activation sequences, hence introducing an additional variable on which the forward dynamics contracts. Let $J_N(\hat{\mathbf{X}})$ be the number of preimages of $\hat{\mathbf{X}}$ and $J_N = \int J_N(\hat{\mathbf{X}}) d\mu(\hat{\mathbf{X}})$ the averaged number then:

$$-\sum_{i=1}^N \lambda_i \sigma_i = \log N - \log J_N \tag{18}$$

III. DYNAMICS AND SOC

A. The Zhang Model as an Iterated Function System

If the system is conjugate to a subshift of finite type, the dynamics of the Zhang model is essentially equivalent to a graph probabilistic Iterated Function System (IFS),^(5, 16) namely, a set of quasi-contractions F_i randomly composed along a Markov graph admitting a unique invariant measure μ^* . Note that IFS are usually defined for true contractions,

however, in our case, any finite composition along the graph is a contraction. In this case, the classical theory of graph directed Iterated Functions Systems applies and allows one to obtain interesting results with respect to the geometrical structure of the invariant set.

1. The Zhang Model Attractor. The IFS determines a unique non-empty compact set \mathcal{A} , called *the attractor of the IFS*, satisfying:

$$\mathcal{A} = \mathcal{F}(\mathcal{A}) \stackrel{\text{def}}{=} \bigcup_{i=1}^{\mathcal{R}_N} F_i(\mathcal{A}) \quad (19)$$

This set is usually a fractal.

Let $\mathbb{H}(\mathcal{M})$ be the set of compact subsets in \mathcal{M} . Define a distance on $\mathbb{H}(\mathcal{M})$, called the *Hausdorff metric* by:

$$\delta(A, B) = \sup\{d(a, B), d(b, A), a \in A, b \in B\} \quad (20)$$

where A, B are non empty closed bounded subsets of \mathcal{M} , and $d(x, A) = \inf\{d(x, a), a \in A\}$. \mathcal{A} is an attractor of the IFS in sense that it satisfies the following property:⁽¹⁸⁾

$$\forall \mathcal{B} \in \mathbb{H}(\mathcal{M}), \mathcal{F}^n(\mathcal{B}) \rightarrow \mathcal{A}$$

in the Hausdorff metric when $n \rightarrow \infty$. Furthermore, if $\mathcal{B} \in \mathbb{H}(\mathcal{M})$ is such, that for all i , $F_i(\mathcal{B}) \subset \mathcal{B}$ then:

$$\mathcal{A} = \bigcap_{n=0}^{\infty} \mathcal{F}^n(\mathcal{B})$$

Therefore, the asymptotics dynamics of the Zhang model lives onto an attractor, further on denoted by \mathcal{A} , whose fractal geometry is linked to the critical behaviour at stationarity. Note however, that, despite one might expect from the presence of dissipation the existence of an attractor with a fractal structure in general SOC models, this is not the case because the distribution of energy has to be of type like in the Zhang model to get local contraction effects. We give now two simple examples of attractors which can be constructed “by hand.”

2. One-Dimensional Chain with $E_c=1$, $N=3$, $\epsilon=0$. For $N=3$, each configuration \mathbf{X} is a triplet $\{X_1, X_2, X_3\}$. First note that only the mappings whose image intersect the cube $[E_c/2, E_c]^3$ are relevant for the asymptotic dynamics. Moreover, for $E_c \leq 1$ each activation generates an avalanche, and the resulting configuration always contains a site with

zero energy. This is an effect of projection onto the complementary set of the kernel of the product mapping, discussed in Section C2.

The mappings (rather, their projection onto the faces of \mathcal{M}) are respectively:

$$\begin{aligned}
 F_1 &= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{8} \end{bmatrix}, & F_2 &= \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \\
 F_3 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \\
 F_4 &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, & F_5 &= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \\
 F_6 &= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \\
 F_7 &= \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}, & F_8 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix} \\
 F_9 &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \frac{1}{8} \\ \frac{1}{4} \end{bmatrix}
 \end{aligned}$$

The Markov transition graph can be easily computed. Each legal transition occurs with probability $\frac{1}{3}$ (activation of sites 1, 2, 3). To obtain the invariant set of the IFS, one must first notice that the three mappings F_3 , F_5 , F_7 have a zero eigenvalue and project vectors in \mathbb{R}^3 along the direction

$\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. These projection induce a tree structure for the

invariant set. The maps send their domain of continuity onto the segments:

$$\begin{aligned}
 a &= \left\{ \mathbf{X} \in \mathbb{R}^3 \left| \mathbf{X} = \lambda \cdot \begin{bmatrix} 0 \\ 3/4 \\ 1/2 \end{bmatrix} + (1-\lambda) \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \lambda \in [0, 1] \right. \right\} \\
 b &= \left\{ \mathbf{X} \in \mathbb{R}^3 \left| \mathbf{X} = \lambda \cdot \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix} + (1-\lambda) \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \lambda \in [0, 1] \right. \right\} \\
 c &= \left\{ \mathbf{X} \in \mathbb{R}^3 \left| \mathbf{X} = \lambda \cdot \begin{bmatrix} 1/2 \\ 3/4 \\ 0 \end{bmatrix} + (1-\lambda) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \lambda \in [0, 1] \right. \right\}
 \end{aligned}$$

We can generate the invariant set by starting with the set: $a \cup b \cup c$. We show in Fig. 1 the initial branches a, b, c and their image under the five first iterates of the IFS. We have labeled the branches of the tree by their corresponding coding, for the three first iterates. One see, then how the tree structure is generated.

3. One-Dimensional Case where $E_c \in](1+\epsilon)/(1-\epsilon), 2/(1-\epsilon)]$. This case is very atypical in the sense that the attractor is a finite set of points whose components have values $(1+\epsilon)/(1-\epsilon)$. Indeed, let \mathcal{I} be the set where each site has energy $X = (1+\epsilon)/(1-\epsilon)$ but at most one with energy $(2\cdot\epsilon)/(1-\epsilon)$. For $E_c \in](1+\epsilon)/(1-\epsilon), 2/(1-\epsilon)]$ the set \mathcal{I} is the unique invariant set. Moreover, $\forall \mathcal{B} \in \mathcal{M}, \mathcal{T}^n(\mathcal{B}) \rightarrow \mathcal{I}$, in the Hausdorff metric. Note that in this case we do not have invertibility, but each point has exactly N preimages.

This behaviour is somehow pathological as it exists only in this range of E_c value. For higher dimensions, we still do not know if there can be such an atomic invariant set.

B. The Probability Distribution of Avalanche Size

In this part we derive a relation linking the sum of Lyapunov exponents and the probability of avalanche size. Then, we relate the fractal structure of the attractor to the critical exponent values. The basic ingredient is the Ledrappier–Young formula. However, as already said, the existence of a kernel in the standard Zhang model makes the analysis somehow cumbersome. We therefore discuss first the non kernel case ($\epsilon \neq 0$) and comment only briefly on the modifications necessary to handle the limiting case $\epsilon = 0$.

1. Average Contraction Rate. The key result is an exact formula linking the determinant of the basic maps to the total number s of overcritical sites in the corresponding avalanche. Namely we prove the following:

Proposition 4. For $T_a^k \in K_s$ one has

$$\det L_a^k = \epsilon^s \quad (21)$$

Proof. We first show a property about the relative distance of the overcritical sites in an avalanche. Let the avalanche be given by $A = \{A_k\}_{0 \leq k \leq n}$ where A_k is the set of overcritical sites at the k th step in

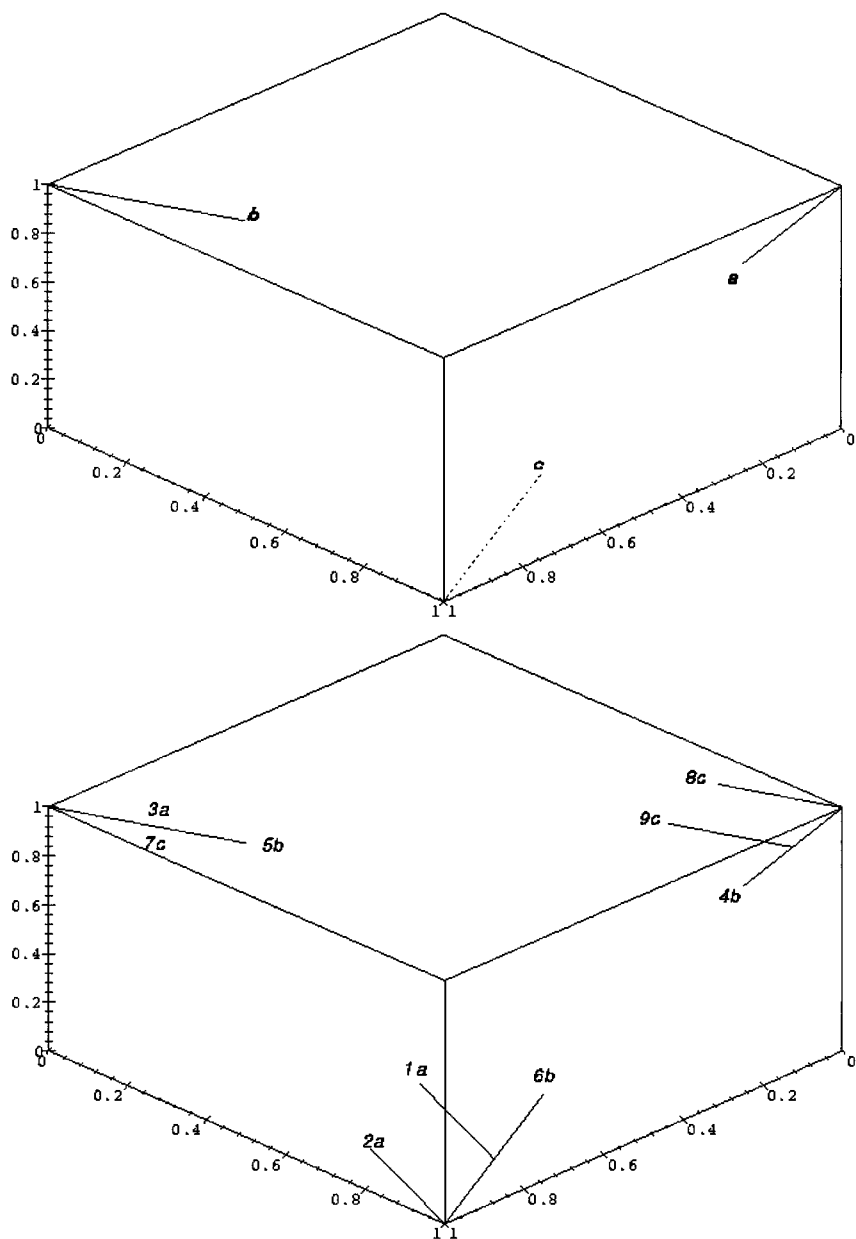


Fig. 1. Five first steps of iteration of the Iterated Function System, for $E_c = 1$, $N = 3$, $\varepsilon = 0$. The labeling of the branches of the tree, is a sequence, read from the left to the right, indicating the sequence of mappings applied to one of the initial branch a , b , c (right most symbol). The last picture is the plot of a trajectory on the attractor.

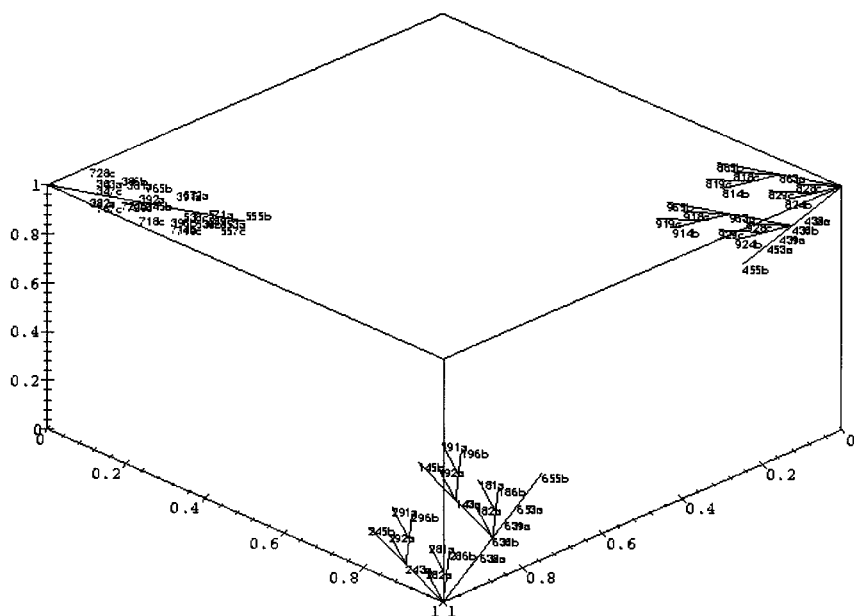
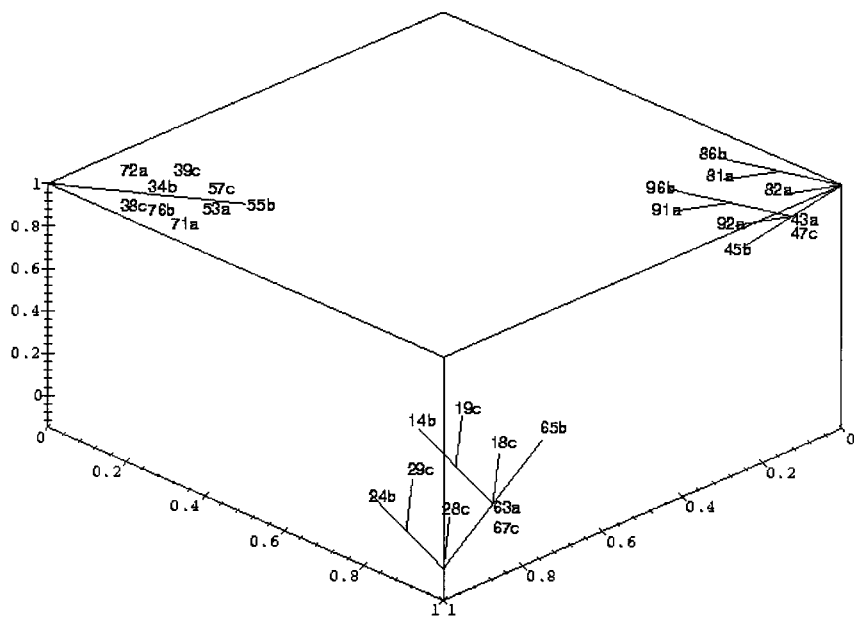


Fig. 1. (Continued)

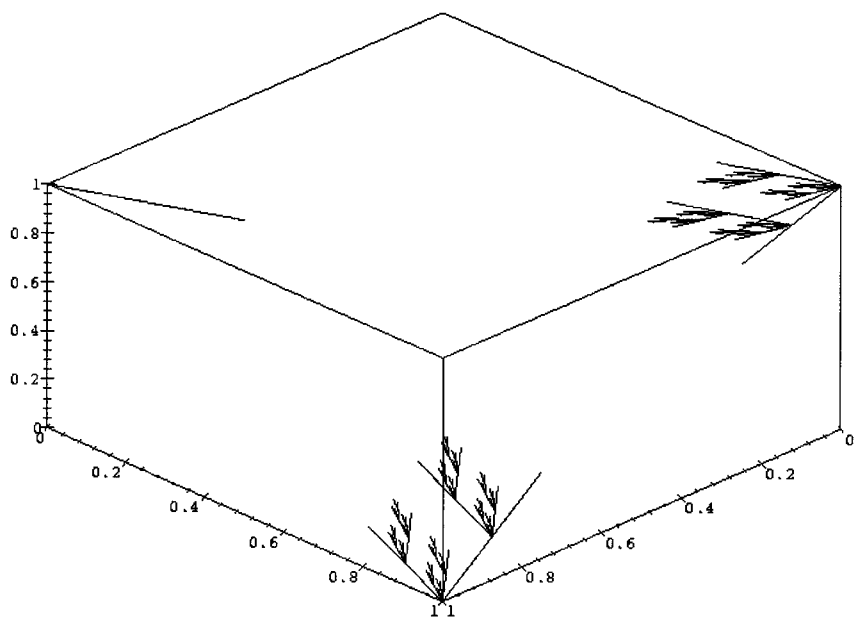
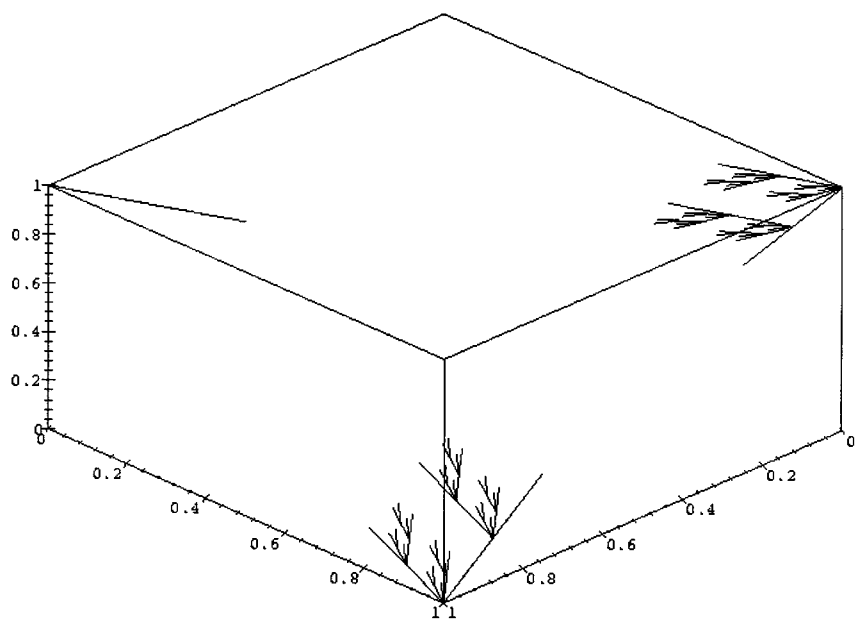


Fig. 1. (Continued)

the avalanche A . Denote by $D(A_k) = \{d(i, j) : i, j \in A_k\}$ the set of pairwise distances of the vertex set A_k . The proof is a straightforward consequence of the following lemma.

Lemma 1. For any \mathbb{Z}^d sublattice A , $D(A_k) \subset 2 \cdot \mathbb{N} \Rightarrow D(A_{k+1}) \subset 2 \cdot \mathbb{N}$.

Let $\gamma(i, j)$ denotes any path from i to j with no repetition of edges and $|\gamma|$ its length. From the general properties of subsets of \mathbb{Z}^d it follows that $d(i, j) \in 2 \cdot \mathbb{N} \Rightarrow |\gamma(i, j)| \in 2 \cdot \mathbb{N}$ and, vice versa, if $|\gamma(i, j)| \in 2 \cdot \mathbb{N}$ for some γ then $d(i, j) \in 2 \cdot \mathbb{N}$. Let $|A_k|, |A_{k+1}| \geq 2$ and $i, j \in A_k$. We will show below, that provided ε is sufficiently small, no site can be overcritical for two successive time steps. Assuming this for the moment, it follows that $A_{k+1} \subset \mathcal{B}(A_k, 1) = \{v \in A : d(v, A_k) = 1\}$ since no site of A_k can be overcritical in the next step. Fix a path γ^* by eliminating the first and last edge of $\gamma \subset A_k$. γ^* is a path between a vertex in $\mathcal{B}(i, 1)$ and $\mathcal{B}(j, 1)$, of length $|\gamma - 2|$. Since the pairwise distance in $\mathcal{B}(v, 1)$ are even for any vertex $v \in A$ it follows that for any pair of vertices from $\mathcal{B}(i, 1)$ to $\mathcal{B}(j, 1)$ there is an even length extension of γ^* connecting those two vertices. This proves the lemma.

We now show that, provided ε is sufficiently small, a site cannot be overcritical in two successive time steps. For $\varepsilon = 0$ this is obvious. Assume now that $\varepsilon > 0$. Let \tilde{E}_k be the maximal energy value of an overcritical site in the k th step of an avalanche. For a given ε we have to show that $\varepsilon \cdot \tilde{E}_k < E_c, \forall k$. Clearly, $\tilde{E}_0 < E_c + 1$. It is obvious that the maximal increase of energy on a site v can only happen if v has $2 \cdot d$ overcritical neighbours. In that case we have the following estimation

$$\tilde{E}_{k+1} < (1 - \varepsilon) \cdot \tilde{E}_k + E_c$$

Iterating this expression we obtain:

$$\tilde{E}_n < (1 - \varepsilon)^n \cdot \tilde{E}_0 + E_c \cdot \sum_{i=0}^{n-1} (1 - \varepsilon)^i = (1 - \varepsilon)^n \cdot \tilde{E}_0 + E_c \cdot \frac{1 - (1 - \varepsilon)^n}{\varepsilon}$$

which has to be less than E_c/ε . This holds provided:

$$E_c > \frac{\varepsilon}{1 - \varepsilon} \quad (22)$$

It follows from the lemma that two neighbours cannot be simultaneously overcritical during one avalanche provided ε sufficiently small. One then gets the expression for the determinant by decomposing the matrix of the avalanche into one step matrices. The row corresponding

to any overcritical site as only one non zero entry, the diagonal element ε (nothing comes from the other overcritical sites at this time) while the columns corresponding to a non overcritical site has only one non zero entry, the diagonal element 1. Formula (21) follows. ■

By using the ergodic theorem we get the log-average volume contraction which is also the sum of Lyapunov exponents as:

$$\sum_i \lambda_i^- = \log \varepsilon \cdot \sum_{s=1}^{S_N} s P_N(s) = \log \varepsilon \cdot \bar{s} \quad (23)$$

where \bar{s} is the average avalanche size and s_N the maximal avalanche size. The formula relates the *local volume contraction* to the *average avalanche size*. It connects therefore *microscopic* dynamical quantities (Lyapunov exponents) to a *macroscopic* observable (average avalanche size). In particular it allows to establish a link between the Lyapunov spectrum and the critical exponents of the avalanche size distribution (see below and ref. 9).

2. Contraction Versus Expansion. The average contraction rate decreases with increasing E_c . Indeed, the larger E_c , the larger is the frequency of occurrence of “trivial” avalanches where no relaxation occurs. They only display neutral directions in the phase space, with no contraction, and no contribution to the negative Lyapunov exponent. This can also be seen on formula (21): the larger E_c , the smaller the average avalanche size. Therefore, for fixed N , there exists an $E_c^*(N)$ which is the unique E_c value such that:

$$\log \varepsilon \cdot \bar{s} + \log(N) = 0 \quad (24)$$

For $E_c < E_c^*(N)$ the contraction dominates the expansion, while it is the opposite for $E_c > E_c^*(N)$. Clearly, the invariant set structure is different in these two cases. On the one hand, for small E_c values, the images of the domains \mathcal{P}_a^k are thin bands which are stretched slower than they contract. Therefore, they are expected not to overlap asymptotically and the invariant set has a Cantor structure with large gaps. On the other hand, when $E_c > E_c^*(N)$, the successive images of the domains \mathcal{P}_a^k fill more and more the phase space and the properties in Conjecture 1 and 4 should not hold.

Note that, in this scheme, the Hausdorff dimension of \mathcal{A} increases for increasing E_c , $E_c < E_c^*(N)$ and is likely to be constant when $E_c > E_c^*(N)$.

The graph of $E_c^*(N)$ can easily be computed numerically. We give an example below, in a square lattice, for various values of ε (Fig. 2). Note

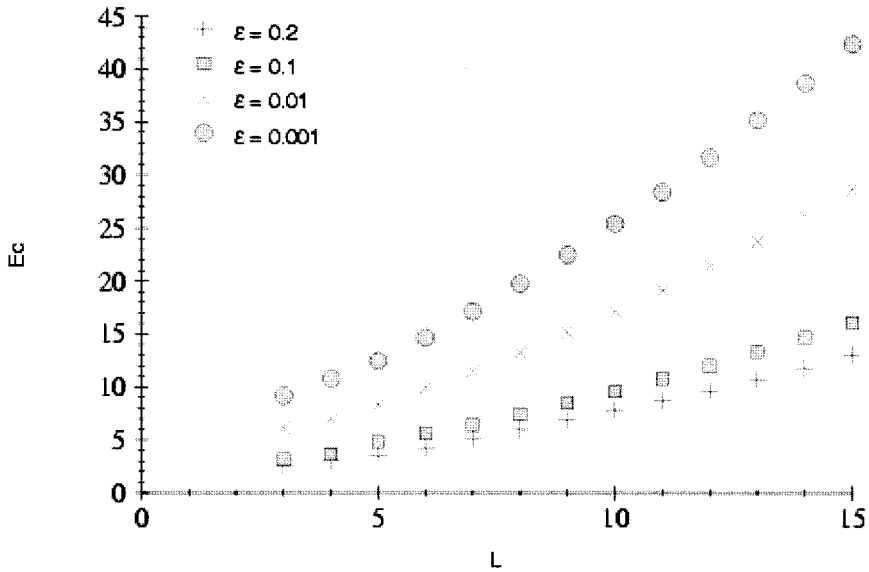


Fig. 2. $E_n^*(N)$ for various values of ϵ . L is here the linear dimension of A ($N = L^2$).

that $E_c^*(N)$ increases with N . Therefore, one expects that, Conjecture 1–4 hold on larger and larger range of E_c values, as N increases.

3. Bounds on the Critical Exponent. In the invertible case the Ledrappier–Young formula implies:

$$\sum_{i=1}^N |\lambda_i| \geq \log N \quad (25)$$

Therefore, from formula (23):

$$\sum_{s=1}^{s_N} s P_N(s) \geq \frac{\log N}{|\log \epsilon|} \quad (26)$$

This implies that the average avalanche size, \bar{s} , has to diverge when N goes to infinity and that, in the thermodynamic limit, $P_N(s)$ tends to a distribution with an infinite mean-value.

Furthermore, a reasonable assumption (supported by experiments) is that, for fixed N , the probability decreases with the avalanche size, namely:

$$\forall N \geq N(\epsilon), \quad P_N(s) \geq P_N(s+1) \quad (27)$$

In a certain way, this behaviour could be expected since the larger the avalanche, the more one has to impose conditions defining the corresponding domain of continuity, and the less the corresponding volume. However, this argument is not completely correct in general since one assumes some kind of absolute continuity of the invariant measure on the stable foliation (the probability of a domain decreases with its volume). In particular it is completely false for $E_c > 1$ in the one dimensional chain, here the probability *increases* with s .

Assuming that there is indeed a power law, and that the system is invertible then one obtains:

$$P_N(s) = \frac{f_N(s)}{s^\tau}, \quad 1 < \tau \leq 2 \quad (28)$$

where $f_N(s)$ is a cut-off function accounting for finite size effects.¹¹

Therefore, Eq. (26) gives *the scaling of the power law and bounds for the critical exponent* τ . In particular, if we assume that $P_N(s)$ converges to some limit $P^*(s)$ as $N \rightarrow +\infty$, then $P^*(s) = c/s^\tau$, $\tau \in [1, 2]$.

4. The Value of τ and the Fractality of the Support of the Invariant Measure. The Ledrappier–Young formula gives a direct way to check the “fractality” of the support of the invariant measure in the invertible case for

$$\left| \sum_i \lambda_i^- \right| > \log N \quad (29)$$

if some partial dimensions σ_i are not integers ($< m_i$).

Suppose that the measure μ is absolutely continuous on the stable foliation. Then from Eq. (17) we get equality in (26), implying that the sum diverges logarithmically with N . Furthermore, the maximal avalanche size scales like $s_N \approx N^{\beta/2}$, implying that $\log s_N \approx \log N$. Then \bar{s} diverges *logarithmically with s_N suggesting a critical exponent $\tau = 2$* . More generally, we get the same result if the fractal set is homogeneous (in sense that all partial Hausdorff dimensions are equal or can be bounded from below as $N \rightarrow \infty$).

However, one does not expect the fractal to be homogeneous. It is indeed clear that the contraction is not uniform in the phase space. For

¹¹ In the SOC literature, the Finite Scaling Assumption leads to write the probability distribution of avalanche size $P_N(s)$ as $P_N(s) = s^{-\tau} \mathcal{G}(s \cdot L^{-\beta})$ where \mathcal{G} accounts for cut-off effects. The exponents (τ, β) are believed to characterize the universality class of the model. Note that τ, β depend a priori from E_c .⁽²¹⁾

$\varepsilon = 0$ the kernel directions produce infinite contraction. In Fig. 1 they are the directions transverse to the “branches” of the attractor, which project the dynamics on the tree, in one time step. On the other hand, the directions “parallel” to the branches produce finite contraction. As a corollary, the partial Hausdorff dimensions of the invariant set are zero transversally to the attractor while they are finite along the branches. When ε is small, there are still directions producing high contractions, those which give the kernel directions as $\varepsilon \rightarrow 0$. This effect is reflected in the Lyapunov spectrum where one detects two parts in the spectrum (see Fig. 3).

Therefore, the sum (resp. \bar{s}) must diverge faster than logarithmically with N and strict inequality holds. This implies that the critical exponent $\tau < 2$. The measured exponent is indeed always strictly lower than 2.⁽²¹⁾ Note that under the finite size scaling hypothesis \bar{s} behaves as $N^{1-(\tau/2)}$ and that τ is indeed lower than 2 iff strict inequality holds in (26).

An explicit formula linking the Hausdorff dimension and the critical exponent (τ, β) can be obtained through the Ledrappier–Young formula. This will be treated in a separated paper.⁽⁹⁾

5. The Case $\varepsilon = 0$. One would like to obtain an equality like (26) also in this case. However, setting $\varepsilon = 0$ leads to a zero determinant and hence infinite Lyapunov exponents (more precisely the Lyapunov exponents

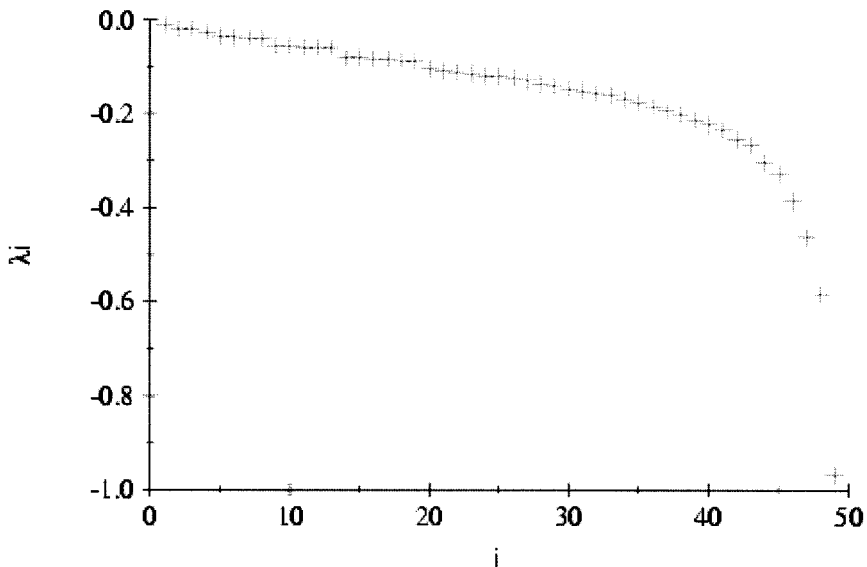


Fig. 3. Spectrum of negative Lyapunov exponents for $N = 49$, $E_c = 3$, $\varepsilon = 0.1$.

are not defined on the whole configuration space). But restricted to $\mathcal{E}^s(\hat{\mathbf{X}})$ one can still compute finite Lyapunov exponents and the sum is just the determinant of the matrix *restricted* to the stable space $\mathcal{E}^s(\hat{\mathbf{X}})$ at some point $\hat{\mathbf{X}}$ in the domain of the map.

If one has a further inequality like:

$$\det(T_a^k|_{\mathcal{E}^s(\hat{\mathbf{x}})}) \geq C^s \quad (30)$$

$\forall K \in K_s$ and $0 < C < 1$ one still gets the same kind of estimates for the expected avalanche size like in the $\varepsilon \neq 0$ case. The details are quite cumbersome and will be given in a forthcoming paper.

C. Phase Transitions

The domains of continuity \mathcal{M}_a^k are bounded by hyperplanes, *which are moving when E_c varies*. In general, a small variation in E_c does not lead to structural changes in the dynamics, if all these hyperplanes are intersecting the interior of \mathcal{M} . In this case, the structure of the transition graph is not modified. Moreover, the corresponding mapping T_a^k does not change under this motion. More precisely, changes in E_c just change the shape of \mathcal{M}_a^k but not the matrix of the mapping T_a^k .

However, for some E_c values, some hyperplanes *have intersection only with $\partial \mathcal{M}$* . This implies that a small change in E_c can push these hyperplanes outside \mathcal{M} . Hence the corresponding transition graph changes in structure. As far as the asymptotic dynamics and therefore, the invariant distribution is dependent on the graph structure, we expect changes in the SOC picture when crossing these *critical* E_c values. This effect has already been reported elsewhere for the one dimensional Zhang model⁽⁷⁾ and arises also in two dimensions where $P_N(s)$ is not a power law for $E_c \ll 1$.⁽⁹⁾

In fact, one can easily figure out that at least the limiting cases $E_c \rightarrow \infty$ and $E_c \rightarrow 0$ are completely different. For $E_c \rightarrow \infty$ relaxation events are more and more seldom. One obtains kind of frozen state where energy increases (on average) monotonously with some rare (but large) avalanches. Moreover, the asymptotic energy distribution is sensitive to the initial conditions (loss of ergodicity). Furthermore, the attractor as a large Hausdorff dimension.

On the other hand, for $E_c \rightarrow 0$, each activation generates a very large avalanche (that has to reflect many times on the boundary before it has lost enough energy to stop). This implies larger and larger contraction, and therefore the sum of Lyapunov exponents decreases to $-\infty$. As a corollary of Ledrappier–Young formula the partial fractal dimensions have to go to zero in order to maintain the product equal to $\log N$.

IV. CONCLUSION

We have shown that certain classes of models of SOC like the Zhang model fit naturally into a well known class of dynamical systems. Especially for the question of asymptotic energy distribution, observables distribution, ergodicity, this seems to be a proper point of view. Furthermore it seems likely to exhibit close relationship between the probability of the size of avalanches and the fractality of the attractor.

There are many questions for further investigations. We list a few of them.

1. Development of a thermodynamic formalism and it's linkage to the SOC quantities. It should be possible to extrapolate this formalism to the case of arbitrary (hyperbolic) SOC-system. Moreover, phase transitions should correspond to changes in the invariant measure of maximal entropy (loss of analyticity of the topological pressure). One expects that in a proper formulation E_c should play the role of an inverse temperature.

2. Dimension spectrum of the attractor and Lyapunov spectrum. We have outlined above the crucial role played by Lyapunov exponents (accounting for energy transport) and the link one can establish with the equilibrium state and the critical exponents. The full developments of this point will be published elsewhere.⁽⁹⁾

3. Nonuniform distribution rules. As outlined in the paper most of our results carry on if one does not choose a uniform activation measure, because one still has a good measure as an equilibrium state. On the other hand, activating with a degenerate probability distribution (for example activating always the same site) will lead to different results. Activating sites periodically with different period will allow to sample the periodic orbits structure of the global attractor, which are dense.

4. Thermodynamic limit for fixed E_c and $N \rightarrow \infty$ and the limit $E_c \rightarrow \infty$ (N fixed). In these both cases one loses the hyperbolic structure.

5. Smooth thresholds. Some modification of Zhang model have been proposed, in particular to treat this model in the continuum limit by an anomalous diffusion equation.^(13,4) In this case the Heaviside function corresponding to the sharp threshold at E_c is smoothed out by some continuous function. The nice effect of this change in our description is that it removes the singularity set. On the other hand, the system is expected to have still a nice hyperbolic structure (though non uniform) where smooth local stable manifolds exist for almost all points. Pesin theory⁽²²⁾ should apply in this context.

6. The case where δX is random. In the usual Zhang model, the energy activation quantum δX is not a constant but is a random variable.

This situation can be treated in the framework of random hyperbolic dynamical system.

As a conclusion we would like to outline that the study of SOC-models with tools from dynamical system theory will certainly not solve all questions in this context. In the belief of the authors it is mainly useful for the study of fairly general structure properties of the models. It is also clear that the complexity of the underlying transition graph on which the model is defined will become of crucial importance for some questions.

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